Deriving a Theoretical Model for Hydrogen Atom

1. In **classical** mechanics, the **centripetal force** acting on an electron in a circular orbit around the nucleus is ${}^{mv^2}\!/_{r}$, where m is the electron's mass, v is its velocity, and r is the orbit's radius.
2. From **Coulomb's Law**, the **force of attraction** between the electron and the nucleus is ${}^{Ze^2}\!/_{r^2}$, where Z is the charge on the nucleus (the atoms atomic number) and e is the electron's elementary charge.
3. To maintain a stable orbit, the electron's centripetal force and its force of attraction to the nucleus must be identical; thus ${}^{mv^2}\!/_{r} = {}^{Ze^2}\!/_{r^2}$ or $mv^2 = {}^{Ze^2}\!/_{r}$. If the centripetal force exceeds the force of attraction, then the electron will escape the atom; if the force of attraction exceeds the centripetal force, then the electron will fall into the nucleus.
4. The electron's total energy, E, is the sum of its kinetic energy, $\frac{1}{2}mv^2$, and its potential energy, $-{}^{Ze^2}\!/_{r}$, or $E = \frac{1}{2}mv^2 - {}^{Ze^2}\!/_{r}$. From step 3, substituting ${}^{Ze^2}\!/_{r}$ for mv^2 gives $E = -{}^{Ze^2}\!/_{2r}$.
5. The classical treatment in steps 1–4 shows us that an electron's energy is a function of its distance, r, from the nucleus; however, there is nothing in this treatment that limits the radius of an electron's orbit or its energy. To **quantize** the atom we assume the electron's angular momentum, $L = mvr$, is limited to integer multiples of ${}^{h}\!/_{2\pi}$ where h is Planck's constant; thus $L = mvr = {}^{nh}\!/_{2\pi}$, where n is a positive integer.
6. From step 3, solving ${}^{mv^2}\!/_{r} = {}^{Ze^2}\!/_{r^2}$ for velocity, v, and substituting into the equation for angular momentum gives $\sqrt{Zmre^2} = {}^{nh}\!/_{2\pi}$. Rearranging and solving for r gives

$$r = \frac{n^2h^2}{4\pi^2 mZe^2}$$

7. The smallest possible radius is for $n = 1$, for which

$$r_1 = \frac{h^2}{4\pi^2 mZe^2}$$

   All other radii are integer multiples of r_1 where $r_n = n^2 r_1$; thus, the allowed orbits and allowed energies for an electron are quantized.
8. To see that the empirical equation for hydrogen's line spectrum emerges from this theoretial model, we substitute $r = {}^{n^2h^2}\!/_{4\pi^2 mZe^2}$ into the equation for the electron's energy, which gives

$$E = -\frac{Ze^2}{2r} = -\frac{2\pi^2 mZ^2e^4}{n^2h^2}$$

   or, after substituting in values for the constants, $E = {}^{-(2.18\times10^{-18})\ \mathrm{J})}\!/_{n^2}$.
9. The change in energy when an electron moves between two orbits, $\Delta E = E_{\text{final}} - E_{\text{initial}}$, is

$$\Delta E = -(2.18\times10^{-18}\ \mathrm{J})\left\{\frac{1}{n^2_{\text{final}}} - \frac{1}{n^2_{\text{initial}}}\right\}$$

10. In terms of wavelength, the equation at step 9 becomes

$$\frac{1}{\lambda} = (1.09737\times10^{7}\ \mathrm{m}^{-1})\left\{\frac{1}{n_1^2} - \frac{1}{n_2^2}\right\}$$

   which is the empirical equation where n_2 is greater than n_1.